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Trigonometric approximation of periodic functions in Morrey spaces using matrix means of their Fourier series

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Abstract. We examine the generalized methods of summability of Fourier series of functions belonging to Morrey spaces $L^{p,\lambda}$, $0 < \lambda \leq 2$, $1 < p < \infty$. In this study, the approximation of functions by matrix means in terms of the continuity modulus in Morrey spaces $L^{p,\lambda}$, $0 < \lambda \leq 2$, $1 < p < \infty$, is investigated.

1. Introduction and main results

Let $\mathbb{T}$ denote the interval $[0, 2\pi]$. Let $L^p(\mathbb{T})$, $1 \leq p < \infty$ be the Lebesgue space of all measurable $2\pi-$periodic functions defined on $\mathbb{T}$ such that

$$\|f\|_p := \left(\int_{\mathbb{T}} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

The *Morrey spaces* $L_0^{p,\lambda}(\mathbb{T})$ for a given $0 \leq \lambda \leq 2$ and $p \geq 1$, we define as the set of functions $f \in L_{loc}^p(\mathbb{T})$ such that

$$\|f\|_{L_0^{p,\lambda}(\mathbb{T})} := \left\{ \sup_I \frac{1}{|I|^{1-\frac{\lambda}{2}}} \int_I |f(t)|^p \, dt \right\}^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all intervals $I \subset [0, 2\pi]$. Note that $L_0^{p,\lambda}(\mathbb{T})$ becomes a Banach spaces, where $\lambda = 2$ coincides with $L^p(\mathbb{T})$ and for $\lambda = 0$ with $L^\infty(\mathbb{T})$. If $0 \leq \lambda_1 \leq \lambda_2 \leq 2$, then $L_0^{p,\lambda_1}(\mathbb{T}) \subset L_0^{p,\lambda_2}(\mathbb{T})$. Also, if $f \in L_0^{p,\lambda}(\mathbb{T})$, then $f \in L^p(\mathbb{T})$ and hence $f \in L^1(\mathbb{T})$. The Morrey spaces were introduced by C.B. Morrey in 1938 [25]. The properties of these spaces have been investigated intensively by several authors and together with weighted

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Lebesgue spaces L^p_ω play an important role in the theory of partial equations in the study of the local behavior of the solutions of elliptic differential equations and describe local regularity more precisely than Lebesgue spaces L^p. The detailed information about properties of the Morrey spaces can be found in references [7–9, 13, 15, 26, 27].

Denote by $C^\infty(\mathbb{T})$ the set of all functions that are realized as the restriction to $\mathbb{T}$ of elements in $C^\infty(\mathbb{T})$. Also we define $L^{p,\lambda}(\mathbb{T})$ to be closure of $C^\infty(\mathbb{T})$ in $L_0^{p,\lambda}(\mathbb{T})$. $L^{p,\lambda}(\mathbb{T})$ is modified Morrey space which contains the set of trigonometric polynomials as a dense subset.

We define Steklov means f_h by

$$f_h(x) := \frac{1}{2h}\int_{-h}^{h} f(x+t)\,dt, \quad 0 < h < \pi,\ x \in \mathbb{T}.$$

According to [9] the inequality

$$\|f_h\|_{L^{p,\lambda}(\mathbb{T})} \leq c\,\|f\|_{L^{p,\lambda}(\mathbb{T})}$$

holds. Hence the operator f_h is bounded in the space $L^{p,\lambda}(\mathbb{T})$, $0 \leq \lambda \leq 2$ and $p \geq 1$.

The function

$$\Omega_{p,\lambda}(\delta, f) := \sup_{|h|\leq\delta} \|f - f_h\|_{L^{p,\lambda}(\mathbb{T})}, \quad \delta > 0$$

is called the *modulus of continuity* of $f \in L^{p,\lambda}(\mathbb{T})$, $0 \leq \lambda \leq 2$ and $p \geq 1$.

The modulus of continuity $\Omega_{p,\lambda}(\delta, f)$ is a nondecreasing, nonnegative, continuous function and

$$\Omega_{p,\lambda}(\delta, f+g) \leq \Omega_{p,\lambda}(\delta, f) + \Omega_{p,\lambda}(\delta, g)$$

for $f, g \in L^{p,\lambda}(\mathbb{T})$, $0 \leq \lambda \leq 2$ and $p \geq 1$.

We define the following class of functions

$$Lip_{p,\lambda}(\omega, M) = \left\{ f \in L^{p,\lambda}(\mathbb{T}) : \Omega_{p,\lambda}(\delta, f) \leq M\omega(\delta),\ \delta > 0 \right\},$$

where ω is a function of modulus of continuity type on the interval $[0, 2\pi]$, i.e., a nondecreasing continious function having the following properties $\omega(0) = 0$, $\omega(\delta_1+\delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1+\delta_2 \leq 2\pi$ and M is some positive constant.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x, f) \tag{1}$$

be Fourier series of the function $f \in L_1(\mathbb{T})$, where

$$A_k(x, f) := (a_k(f)\cos kx + b_k(f)\sin kx),$$

$a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function $f \in L_1(\mathbb{T})$.

The n-th *partial sums* of the series (1) is defined as

$$S_n(x,f)=\frac{a_0}{2}+\sum_{\nu=1}^{n}A_\nu(x,f),$$

The best approximation to $f\in L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$, $1<p<\infty$ in the class $\prod_n$ of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{L^{p,\lambda}(\mathbb{T})}:=\inf\left\{\|f-T_n\|_{L^{p,\lambda}(\mathbb{T})}:T_n\in\prod\nolimits_n\right\}.$$

Let $A:=(a_{n,k})_{0\leq k,n<\infty}$ be an infinite matrix of the real numbers such that

$$a_{n,k}\geq 0,\ \text{ when }\ k,n=0,1,2,\ldots,\ \ \lim_{n\to\infty}a_{n,k}=0\ \text{ and }\ \sum_{k=0}^{\infty}a_{n,k}=1$$

or $A_0:=(a_{n,k})_{0<k,\leq n<\infty}$, where

$$a_{n,k}=0,\ \text{ when }\ k>n,$$

then

$$T_{n,A}(x,f):=\sum_{k=0}^{\infty}a_{n,k}S_k(x,f),\quad n=0,1,2,\ldots$$

or

$$T_{n,A_0}(x,f):=\sum_{k=0}^{n}a_{n,k}S_k(x,f),\quad n=0,1,2,\ldots$$

respectively.

We will use the relation $f=O(g)$, which means that $f\leq cg$ for a constant c independent of f and g.

The approximation of the functions by trigonometric polynomials in non-weighted and weighted Morrey spaces has been studied by several authors (see, for example, [6, 10, 18, 20, 23, 30]). In this study, we investigate the approximation of functions using matrix means in terms of continuity modulus in Morrey spaces $L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$ and $1<p<\infty$. These approximations were first obtained by W. Lenski and B. Szal [24] in Lebesgue spaces with variable exponents $L_{2\pi}^{p(x)}$ with $p(x)\geq 1$. Similar results in different spaces were studied by several authors (see, for example, [1–5, 11, 12, 16, 19, 21, 22, 29, 31]).

Note that in this study, to prove main results, the methods used in [17, 24, 28] have been followed.

Our main results are the following.

Theorem 1. *Let* $f \in L^{p,\lambda}(\mathbb{T})$, $0 < \lambda \leq 2$ *and* $1 < p < \infty$. *If the conditions*

$$\sum_{k=0}^{\infty} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| = O\left(\frac{1}{n+1}\right) \tag{2}$$

for some $\beta \geq 0$ *and*

$$\sum_{k=0}^{\infty} (k+1)\, a_{n,k} = O\,(n+1) \tag{3}$$

hold, then the inequality

$$\|T_{n,A}(\cdot, f) - f\|_{L^{p,\lambda}(\mathbb{T})} = O\left(\Omega_{p,\lambda}(f, \frac{1}{n+1}) + \sum_{k=0}^{n} a_{n,k} \Omega_{p,\lambda}(, f, \frac{1}{k+1})\right)$$

holds.

Theorem 2. *Let* $f \in L^{p,\lambda}(\mathbb{T})$, $0 < \lambda \leq 2$ *and* $1 < p < \infty$. *If the conditions*

$$\sum_{k=0}^{\infty} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k}+1}{(k+2)^{\beta}} \right| = O\,(a_{n,n}) \tag{4}$$

for some $\beta \geq 0$ *and*

$$(n+1)\, a_{n,n} = O\,(1) \tag{5}$$

hold, then the inequality

$$\|T_{n,A_0}(\cdot, f) - f\|_{L^{p,\lambda}(\mathbb{T})} = O\left(\sum_{k=0}^{n} a_{n,k} \Omega_{p,\lambda}(, f, \frac{1}{k+1})\right) \tag{6}$$

holds.

The following theorems hold specifically for the class of functions $Lip_{p,\lambda}(\omega, M)$.

Theorem 3. *Let* $f \in Lip_{p,\lambda}(\omega, M)$. *If the conditions* (2) *for some* $\beta > 0$ *and* (3) *hold, then the estimation*

$$\|T_{n,A}(\cdot, f) - f\|_{L^{p,\lambda}(\mathbb{T})} = O\left(\omega\Big(\frac{1}{n+1}\Big)\right) \tag{7}$$

holds.

Theorem 4. *Let* $f \in Lip_{p,\lambda}(\omega, M)$. *If the conditions* (4) *for some* $\beta > 0$ *and* (5) *hold, then the estimation*

$$\|T_{n,A_0}(\cdot, f) - f\|_{L^{p,\lambda}(\mathbb{T})} = O\left(\omega\Big(\frac{1}{n+1}\Big)\right)$$

holds.

Note that $a_{n,k} = e^{-n}\sum_{j=k}^{\infty}\frac{n^j}{(j+1)!}$ where $n,k = 0,1,2,\ldots$, satisfies the condition (2) for any $\beta \geq 0$ and (3). Also $a_{n,k} = \frac{(k+1)^{\beta}-k^{\beta}}{(n+1)^{\beta}}$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$, where $n,k = 0,1,2,\ldots$, satisfies the conditions (4) for any $\beta > 1$ and (5). (see, [24, Example 1 and Example 2]).

2. Auxiliary results

In the proof of the main results we use the following auxiliary results.

Lemma 1 ([32]). *Let $L^{p,\lambda}(T)$ be a Morrey space with $0 < \lambda \leq 2$ and $1 < p < \infty$, $f(x,y)$ is measurable on $\mathbb{R}^2$ and 2π periodic in each variable. Then*

$$\left\|\int_{\mathbb{T}} |f(\cdot,y)|\,dy\right\|_{L^{p,\lambda}(\mathbb{T})} \leq \int_{\mathbb{T}} \|f(\cdot,y)\|_{L^{p,\lambda}(\mathbb{T})}\,dy.$$

Lemma 2 ([24]). *If (2) for some $\beta \geq 0$ and (3) hold, then*

$$\frac{1}{2\pi}\int_{\mathbb{T}}\left|\sum_{k=0}^{\infty} a_{n,k}\sum_{m=0}^{m} l_m \cos mt\right| dt = O(1),$$

where

$$l_m = \begin{cases} 1, & \text{when } m = 0; \\ \frac{\pi}{4\sin\frac{\pi}{8}}, & \text{when } m > 0. \end{cases}$$

Lemma 3. *Let $L^{p,\lambda}(T)$ be a Morrey space with $0 < \lambda \leq 2$ and $1 < p < \infty$. Then*

$$\left\|f_{\frac{1}{2s}}(\cdot+\tau)\right\|_{L^{p,\lambda}(\mathbb{T})} = O(1)\,\|f(\cdot)\|_{L^{p,\lambda}(\mathbb{T})}$$

for every real τ, where $f_{\frac{1}{2s}}(+\tau) = s\int_{-\frac{1}{2s}+\tau}^{\frac{1}{2s}+\tau} f(t)\,dt$ with $s > 1$.

Proof. Let $I = [a,b] \subset [0,2\pi]$ and $b - a \leq 2\pi$. Using the generalized Minkowskii inequality we have

$$\left\{\frac{1}{|I|^{1-\frac{\lambda}{2}}}\int_I \left|f_{\frac{1}{2s}}(x+\tau)\right|^p dx\right\}^{\frac{1}{p}} = \left\{\frac{1}{|I|^{1-\frac{\lambda}{2}}}\int_I \left|s\int_{-\frac{1}{2s}+\tau}^{\frac{1}{2s}+\tau} f(t)\right|^p dt\right\}^{\frac{1}{p}}$$

$$\leq \frac{1}{|I|^{\left(1-\frac{\lambda}{2}\right)/p}} s\int_{-\frac{1}{2s}+\tau}^{\frac{1}{2s}+\tau}\left(\int_I |f(t)|^p\,dt\right)^{\frac{1}{p}} dx. \tag{8}$$

Taking the supremum in the left-hand side of (8) over I we obtain the inequality of Lemma 3. □

Lemma 4. *Let $L^{p,\lambda}(T)$ be a Morrey space with $0<\lambda\leq 2$, $1<p<\infty$ and T_n be a trigonometric polynomial of the degree at most n, such that $\|f-T_n\|_{L^{p,\lambda}(\mathbb{T})}=O(1)\,\Omega_{p,\lambda}(,f,\frac{1}{n+1})$. If (2) for some $\beta\geq 0$ and (3) hold, then the estimation*

$$\left\|\sum_{k=0}^{\infty}a_{n,k}S_k(\cdot,f-T_n)\right\|_{L^{p,\lambda}(\mathbb{T})}=O\left(\Omega_{p,\lambda}(,f,\frac{1}{n+1})\right)$$

holds.

Proof. We define $f_h(t)=\frac{1}{2h}\int_{-h}^{h}f(y+t)\,dy$ and $T_{n,h}(t)=\frac{1}{2h}\int_{-h}^{h}T_n(y+t)\,dy$. If the calculations in the study [24] are taken into account, we have

$$\begin{aligned}&\sum_{k=0}^{\infty}a_{n,k}S_k(x,f-T_n)\\ &=\frac{1}{\pi}\int_{\mathbb{T}}(f_h(t+x)-T_{n,h}(t+x))\sum_{k=0}^{\infty}a_{n,k}\left(\frac{1}{2}+\sum_{m=0}^{k}\frac{mh}{\sin mh}\cos mt\right).\end{aligned}\tag{9}$$

Let $0<h<\frac{1}{2}$ and $|t|\leq\pi$ be given. Using (9), Lemma 1 and Lemma 3 we find that

$$\begin{aligned}&\left\|\sum_{k=0}^{\infty}a_{n,k}S_k(\cdot,f-T_n)\right\|_{L^{p,\lambda}(\mathbb{T})}\\ &\leq\frac{2}{\pi}\int_{\mathbb{T}}\|f_{h_h}(t+\cdot)-T_{n,h}(t+\cdot)\|_{L^{p,\lambda}(\mathbb{T})}\left|\sum_{k=0}^{\infty}a_{n,k}\left(\frac{1}{2}+\sum_{m=0}^{k}\frac{mh}{\sin mh}\cos mt\right)\right|dt\\ &=O(1)\frac{1}{\pi}\int_{\mathbb{T}}\|f-T_n\|_{L^{p,\lambda}(\mathbb{T})}\left|\sum_{k=0}^{\infty}a_{n,k}\left(\frac{1}{2}+\sum_{m=0}^{k}\frac{mh}{\sin mh}\cos mt\right)\right|dt.\end{aligned}$$

In the last estimation, if $h=\frac{\pi}{8m}<\frac{1}{2}$ is taken for $m=1,2,\ldots,$ we conclude that

$$\begin{aligned}&\left\|\sum_{k=0}^{\infty}a_{n,k}S_k(\cdot,f-T_n)\right\|_{L^{p,\lambda}(\mathbb{T})}\\ &\leq O(1)\frac{1}{2\pi}\int_{\mathbb{T}}\left|\sum_{k=0}^{\infty}a_{n,k}\left(1+\frac{\pi}{4\sin\frac{\pi}{8}}\sum_{m=1}^{k}\cos mt\right)dt\right|\|f-T_n\|_{L^{p,\lambda}(\mathbb{T})}.\end{aligned}\tag{10}$$

By considering of Lemma 2

$$\frac{1}{2\pi}\int_{\mathbb{T}}\left|\sum_{k=0}^{\infty}a_{n,k}\left(1+\frac{\pi}{4\sin\frac{\pi}{8}}\sum_{m=1}^{k}\cos mt\right)dt\right|=O(1).\tag{11}$$

is obtained.

The consideration of (10) and (11) gives us

$$\left\| \sum_{k=0}^{\infty} a_{n,k} S_k\left(\cdot, f - T_n\right) \right\|_{L^{p,\lambda}(\mathbb{T})} = O\left(1\right) \left\| f - T_n \right\|_{L^{p,\lambda}(\mathbb{T})}$$
$$= O\left(1\right) \Omega_{p,\lambda}\big(, f, \frac{1}{n+1}\big).$$

The proof of Lemma 4 is completed. □

3. Proofs of the main results

Proof of Theorem 1. Let $f \in L^{p,\lambda}\left(\mathbb{T}\right)$, $0 < \lambda \leq 2$, $1 < p < \infty$ and T_n be the polynomial satisfying the condition $\| f - T_n \|_{L^{p,\lambda}(\mathbb{T})} = \Omega_{p,\lambda}(, f, \frac{1}{n+1})$. It is clear that

$$S_k\left(x, f - T_n\right) = \begin{cases} S_k\left(x, f\right) - T_k\left(x\right), & \text{for } k \leq n \\ S_k\left(x, f\right) - T_n\left(x\right), & \text{for } k \geq n \end{cases} \tag{12}$$

Then taking account of (12) we have

$$\begin{aligned} &\left\| T_{n,A}\left(\cdot, f\right) - f \right\|_{L^{p,\lambda}(\mathbb{T})} \\ &= \Bigg\| T_{n,A}\left(\cdot, f\right) - \sum_{k=0}^{n} a_{n,k} T_k - \sum_{k=n+1}^{\infty} a_{n,k} T_n \\ &\quad + \sum_{k=0}^{n} a_{n,k} T_k + \sum_{k=n+1}^{\infty} a_{n,k} T_n - f \Bigg\|_{L^{p,\lambda}(\mathbb{T})} \\ &\leq \left\| T_{n,A}\left(\cdot, f\right) - \sum_{k=0}^{n} a_{n,k} T_k - \sum_{k=n+1}^{\infty} a_{n,k} T_n \right\|_{L^{p,\lambda}(\mathbb{T})} \\ &\quad + \left\| \sum_{k=0}^{n} a_{n,k} T_k - \sum_{k=n+1}^{\infty} a_{n,k} T_n - f \right\|_{L^{p,\lambda}(\mathbb{T})} \\ &= \left\| \sum_{k=0}^{n} a_{n,k} \left\{ S_k\left(., f\right) - T_k \right\} + \sum_{k=n+1}^{\infty} a_{n,k} \left\{ S_k\left(., f\right) - T_n \right\} \right\|_{L^{p,\lambda}(\mathbb{T})} \\ &\quad + \left\| \sum_{k=0}^{n} a_{n,k} \left(f - T_k\right) + \sum_{k=n+1}^{\infty} a_{n,k} \left(f - T_n\right) \right\|_{L^{p,\lambda}(\mathbb{T})} \\ &\leq \left\| \sum_{k=0}^{\infty} a_{n,k} S_k\left(., f - T_n\right) \right\|_{L^{p,\lambda}(\mathbb{T})} + \left\| \sum_{k=0}^{n} a_{n,k} \left(f - T_k\right) \right\|_{L^{p,\lambda}(\mathbb{T})} \\ &\quad + \left\| \sum_{k=n+1}^{\infty} a_{n,k} \left(f - T_n\right) \right\|_{L^{p,\lambda}(\mathbb{T})} \end{aligned} \tag{13}$$

$$= \left\| \sum_{k=0}^{\infty} a_{n,k} S_k\left(.,f-T_n\right) \right\|_{L^{p,\lambda}(\mathbb{T})} + O\left(1\right) \sum_{k=0}^{n} a_{n,k} \Omega_{p,\lambda}\Big(, f, \frac{1}{k+1}\Big)$$
$$+ \sum_{k=n+1}^{\infty} a_{n,k} \Omega_{p,\lambda}\Big(, f, \frac{1}{n+1}\Big).$$

By using of Lemma 4 we have

$$\left\| \sum_{k=0}^{\infty} a_{n,k} S_k\left(.,f-T_n\right) \right\|_{L^{p,\lambda}(\mathbb{T})} = O\left(\Omega_{p,\lambda}\Big(, f, \frac{1}{n+1}\Big) \right) \tag{14}$$

By combininig (13) and (14) we obtain the inequality of Theorem 1. □

Proof of Theorem 2. If we take into account the definition of matrix A_0, we can write the following estimation [24]:

$$\sum_{k=0}^{\infty} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right|$$
$$= \sum_{k=0}^{n-1} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| + a_{n,n}$$
$$= O\left(a_{n,n}\right) + a_{n,n} = O\left(a_{n,n}\right) = O\left(\frac{1}{n+1}\right)$$

and

$$\sum_{k=0}^{\infty} (k+1)\, a_{n,k} = \sum_{k=0}^{n} (k+1)\, a_{n,k} \leq (n+1) \sum_{k=0}^{n} a_{n,k} = n+1.$$

Using the monotonicity of $\Omega_{p,\lambda}(f,\delta)$ for $\delta > 0$ and $\sum_{k=0}^{n} a_{n,k} = 1$ we obtain

$$\sum_{k=0}^{n} a_{n,k} \Omega_{p,\lambda}(, f, \frac{1}{k+1}) \geq \Omega_{p,\lambda}\Big(, f, \frac{1}{n+1}\Big) \sum_{k=0}^{n} a_{n,k}$$
$$= \Omega_{p,\lambda}\Big(, f, \frac{1}{n+1}\Big).$$

The last inequality and Theorem 1 imply that (6).

The proof of Theorem 2 is completed. □

Proof of Theorem 3. For the function ω of modulus of continuity type the inequality $\omega(n\delta) \le n\omega(\delta)$ holds because of the inequality $\omega(s\delta) \le (s+1)\,\omega(\delta)$, where $n \in \mathbb{N}$ and $s \ge 0$. Then

$$\begin{aligned}\omega(\delta_2) &= \omega\left(\frac{\delta_1}{\delta_1}\delta_2\right) \le \omega\left(\frac{\delta_2}{\delta_1}+1\right)\omega(\delta_1)\\ &= \omega\left(\frac{\delta_2}{\delta_1}+\frac{\delta_1}{\delta_1}\right)\omega(\delta_1) \le \omega\left(\frac{\delta_2}{\delta_1}+\frac{\delta_2}{\delta_1}\right)\omega(\delta_1) = 2\frac{\delta_2}{\delta_1}\omega(\delta_1)\end{aligned}$$

holds, where $0 < \delta_1 \le \delta_2$. Then, from the last inequality we conclude that [24]

$$\frac{\omega(\delta_2)}{\delta_2} \le \frac{2\omega(\delta_1)}{\delta_1}, \quad 0 < \delta_1 \le \delta_2$$

If we use the last inequality and (2) for $\beta > 0$, we get

$$\begin{aligned}&\sum_{k=0}^{n} a_{n,k}\omega\left(\frac{1}{k+1}\right) = \sum_{k=0}^{n} \frac{a_{n,k}}{k+1}\frac{\omega\left(\frac{1}{k+1}\right)}{\frac{1}{k+1}}\\ &\le 2(n+1)\,\omega\left(\frac{1}{n+1}\right)\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\\ &= 2(n+1)\,\omega\left(\frac{1}{n+1}\right)\sum_{k=0}^{\infty}(k+1)^{\beta-1}\frac{a_{n,k}}{(k+1)^{\beta}}\\ &\le 2(n+1)\,\omega\left(\frac{1}{n+1}\right)\sum_{k=0}^{\infty}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right|\sum_{m=0}^{k}(m+1)^{\beta}\\ &= O(n+1)\,\omega\left(\frac{1}{n+1}\right)\sum_{k=0}^{\infty}(k+1)^{\beta}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right|\\ &= O(n+1)\,\omega\left(\frac{1}{n+1}\right)O\left(\frac{1}{n+1}\right) = O\left(\omega\left(\frac{1}{n+1}\right)\right).\end{aligned}$$

This relation and Theorem 1 immediately yield (7).
The proof of Theorem 3 is completed. □

Using Theorem 2, the proof of Theorem 4 is similar to the proof of Theorem 3.

4. Conclusion

Morrey spaces were introduced by Morrey [25] in 1938 in connection with certain problems in elliptic partial defferential equations and calculus of variations. Later, Morrey spaces have played an important role in applications related to the Navier Stokes and Schrödinger equations, as well as elliptic problems with discontinuous coefficients and potential theory. For this reason, it is important to study the problems of approximation theory of functions within Morrey spaces.We investigate the approximation of functions by matrix means in terms of continuity modulus in Morrey spaces $L^{p,\lambda}(\mathbb{T})$, with $0 < \lambda \leq 2$ and $1 < p < \infty$. We consider the general methods of summablity of Fourier series of functions from Morrey spaces $L^{p,\lambda}(T)$ with $0 < \lambda \leq 2$ and $1 < p < \infty$. Note that for the estimating of the error of approximation of functions by the matrix means we use a modulus of continuity constructed by the Steklov functions.

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References

[1] F. G. Abdullayev, S. O, Chaichenko, M. Imash Kyzy, A. Shidlich, *Direct and inverse approximation theorems in weighted Orlicz type spaces, with a variable exponent*, Turkish Journal of Mathematics, 44 (1) (2020), 284-299.

[2] F. G. Abdullayev, S. O, Chaichenko, A. Shidlich, *Direct and inverse approximation theorems of functions in the Musielak-Orlicz type spaces*, Mathematical Inequalities & Applications, 24 (2) (2021), 323-336.

[3] F. G. Abdullayev, S. O. Chaichenko, M. Imashkyzy, A.L. Shidlich, *Jackson-type inequalities and widths of functional classes in the Musielak-Orlicz type spaces*, Rocky Mountain Journal of Mathematics, 51 (4) (2021), 1143-1155.

[4] R. Akgun, *Jackson and converse theorems of trigonometric approximation in weighted Lebesgue spaces*, Proceedengs of A. Razmadze Mathematical Institute, (2010), 1-18.

[5] R. Akgun, D. M. Israfilov, *Approximation and moduli of fractional orders in Smirnov-Orlicz classes*, Glasnik Matematički, 43 (1) (2008), 121-136.

[6] A. Almeida, S. Samko, *Appropximation in generalized Morrey spaces*, Georgian Mathematical Journal, 25 (2) (2018), 155-168.

[7] B. T. Bilalov, F. Seyidova, *Basicity of a system of exponents with a piecewise linear phase Morrey-type spaces*, Turkish Journal of Mathematics, 43 (2019), 1850-1866.

[8] B. T. Bilalov, A.A. Huseynli, S. R. El-Shabrawy, *Basis properties of trigonometric systems in weighted Morrey spaces*, Azerbaijan Journal of Mathematics, 9 (2) (2019), 166-192.

[9] R. R. Coifman, R. Rocherbeg, *Another characterization of BMO*, Proceedings of the American Mathematical Society, 79 (2) (1980), 249-254.

[10] Z. Cakir, C. Akyol, D. Soylemez, A. Serbetci, *Approximation by trigonometric polynomials in weighted Morrey spaces*, Tbilisi Mathematical Journal, 13 (1) (2020), 123-138.

[11] U. Deger, I. Dagadur and M. Kücükaslan, *Approximation by trigonometric polynomials to functions in L_p norm*, Proceedings of the Jangjeon Mathematical Society, 15 (2) (2012), 203-213.

[12] U. Deger, M. Kaya, *On the approximation by Cesaro submethod*, Palestine Journal of Mathematics, 4 (1) (2015), 44-56.

[13] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Non-linear Elliptic Systems*, Princeton University Press, Princeton, 1983.

[14] V. S. Guliyev, A. Ghorbanalizadeh, Y. Sawano, *Approximation by trigonometric polynomials in variable exponent Morrey spaces*, Analysis and Mathematical Physics, 9 (3) (2018), 1265-1285.

[15] F. A. Guliyeva, S.R. Sadigova, *On some properties of convolution in Morrey-type spaces*, Azerbaijan Journal of Mathematics, 8 (1) (2018), 140-150.

[16] A. Guven, D. M. Israfilov, *Approximation by means of Fourier trigonometric series in weighted Orlicz spaces*, Advanced Studies in Contemporary Mathematics (Kyundshang), 19 (2) (2009), 283-295.

[17] A. Guven, D. M. Israfilov, *Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$*, Journal of Mathematical Inequalities, 4 (2) (2010), 285-299.

[18] D. M. Israfilov, N. P. Tozman, *Approximation in Morrey-Smirnov classes*, Azerbaijan Journal of Mathematics, 1 (2) (2011), 99-113.

[19] S. Z. Jafarov, *Approximation by Fejér sums of Fourier trigonometric series in weighted Orlicz spaces*, Hacettepe Journal of Mathematics and Statistics, 42 (3) (2013), 259-268.

[20] S. Z. Jafarov, *Estimates of the approximation by Zygmund sums in Morrey-Smirnov classes of analytic functions*, Azerbaijan Journal of Mathematics, 10 (2) (2020), 109-123.

[21] X. Z. Krasniqi X. Z, *Trigonometric approximation of (signals) functions by Nörlund type means in the variable space $L^{p(x)}$*, Palestine Journal of Mathematics, 6 (1) (2017), 84-93.

[22] X. Z, Krasniqi, *On trigonometric approximation of continuous functions by deferred matrix means*, Australian Journal of Mathematical Analysis and Applications, 19 (1) (2022), Article ID: 3, 1-14.

[23] A. Kinj, M. Ali and S. Mahmoud, *Approximation properties of de la Vallée- Poussin sums in Morrey spaces*, Sultan Qaboos University Journal for Science (SQUJS), 22 (2) (2017), 89-95.

[24] W. Łenski, B.Szal, *Trigonometric approximation of functions from $L_{2\pi}^{p(x)}$*, Results in Mathematics, 75 (2020), Article ID: 56, 1-14.

[25] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Transactions of the American Mathematical Society, 43 (1938), 126-166.

[26] A. M. Najafov, *Some properties of functions from the intercestion of Besov-Morrey type spaces with dominant mixed derivatives*, Proceedengs of A. Razmadze Mathematical Institute, 139 (2005), 71-82.

[27] A. M. Najafov, *The deferentiad properties of functions from Sobolev-Morrey type spaces of functional order*, Journal of Mathematics Research, 7 (3) (2005), 149-158.

[28] T. N. Shakh-Emirov, *Approximation properties of some types of linear means in space $L_{2\pi}^{p(x)}$*, Izv. Sarat. Inst. New ser. Ser. Mat. Mech. Inf., 1 (2) (2013), 108-112.

[29] A. Testici, D. M. Israfilov, *Linear methods of approximation in weighted Lebesgue spaces with variable exponent*, Hacettepe Journal of Mathematics and Statistics, 50 (3) (2021), 744-753.

[30] A. Testici, D.M. Israfilov, *Approximation by matrix transforms in Morrey spaces*, Problemy Analiza –Issues of Analysis, 10 (28) (2) (2021), 79-98.

[31] S.S. Volosivets, *Approximation by linear means of Fourier series and realization functions in weighted Orlicz spaces*, Problemy Analiza –Issues of Analysis, 11 (29) (2) (2022), 106-108.

[32] S. S. Volosivets, *Riesz-Zygmund means and trigonometric approximation in Morrey spaces*, The Journal of Analysis, 32 (2024), 3277-3296.

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